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HEADQUARTERS
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SESSION A-V-3

TITLE: Behavior of Non-Newtonian Fluids Under Conditions of High Pressure, Rapid Acceleration and High Local Velocity

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ABSTRACT: An investigation of the basic relationships which describe the flow of non-Newtonian fluids under conditions of impact loading, possible plastic flow, and short periods for the various conditions.

A generalization of the Navier-Stokes equation is proposed for non-Newtonian fluids in which the stress is proportional to a power n of the rate of strain. Results show that the ratio of the transverse to the longitudinal strain is not a constant but a function of time or of strain and depends strongly upon the visco-elastic response of the medium as well as on the loading conditions. The pronounced effect of volume-viscosity on the shape of this function is illustrated.

SESSION A-V-4

BEHAVIOR OF NON-NEWTONIAN FLUIDS UNDER CONDITIONS OF HIGH PRESSURE RAPID ACCELERATION AND HIGH VELOCITY

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Pseudo Plastic and Dilatant Fluids

The simple shear flow of pseudo-plastic and dilatant fluids is expressed with considerable exactness by Waele-Ostwald's formula as follows:

$$\frac{du}{dy} = \frac{\tau^n}{\mu_{psu}} \qquad (1)$$

where μ_{psu} is the quantity similar to the viscosity of Newtonian fluid, τ is the shearing stress, du/dy is the velocity gradient, n is the rheological constant. The flow of pseudo-plastic fluids is expressed by $n > 1$, and that of dilatant fluids is given by $n < 1$ in the above equation, respectively.

The relation between the stress components and rate-of-strain components for non-Newtonian fluids can be expressed as:

$$p_{ik} = \eta e_{ik} - p\delta_{ik} \qquad (2)$$

Where p_{ik} is a component of the stress tensor (p_{ik}), as indicated in the following array:

$$(p_{ik}) = \begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{bmatrix} \qquad (3)$$

and e_{ik} is a component of the rate-of-strain tensor (e_{ik}) which is expressed as:

$$(e_{ik}) = \begin{bmatrix} e_{xx} & \frac{1}{2}e_{xy} & \frac{1}{2}e_{xz} \\ \frac{1}{2}e_{yx} & e_{yy} & \frac{1}{2}e_{yz} \\ \frac{1}{2}e_{zx} & \frac{1}{2}e_{zy} & e_{zz} \end{bmatrix} \qquad (4)$$

These tensors are symmetric tensors, that is:

$$p_{ik} = p_{ki}\,, \quad e_{ik} = e_{ki} \qquad (i \neq k)$$

p_{ik} is a normal stress if $i = k$, a tangential or a shear stress if $i \neq k$, p is the pressure, which is the mean of the normal stresses over three planes mutually at right angles. δ_{ik} is the Kronecker delta ($\delta_{ik} = 1$ if $i=k$, $\delta_{ik} = 0$ if $i \neq k$; $i,k = x,y,z$). u, v, w are parallel velocity components to the coordinate axes of rectangular Cartesian coordinates x, y, z, respectively. e_{xx}, e_{yy}, etc. are

$$e_{xx} = \frac{\Delta u}{\Delta x}\,, \quad e_{xy} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y} \quad \text{etc. } (\Delta \text{ denotes partial derivative})$$

η is an arbitrary scalar function of the three invariants of the rate-of-strain, and may be expressed as

$$\Theta = e_{xx} + e_{yy} + e_{zz} \qquad (5)$$

$$e = e_{yy}e_{zz} + e_{zz}e_{xx} + e_{xx}e_{yy} - \tfrac{1}{4}(e_{yz}^2 + e_{zx}^2 + e_{xy}^2) \qquad (6)$$

$$|e_{ik}| = e_{xx}e_{yy}e_{zz} + \tfrac{1}{4}\, e_{yz}e_{zx}e_{xy} - \tfrac{1}{4}(e_{xx}e_{yz}^2 + e_{yy}e_{zx}^2 + e_{zz}e_{xy}^2) \qquad (7)$$

If we treat the fluid as being incompressible, the first invariant Θ vanishes, or

$$e_{xx} + e_{yy} + e_{zz} = 0 \qquad (8)$$

Consequently, η can be expressed as an arbitrary scalar function of the two invariants e, $|e_{ik}|$. In rectilinear flow and in two-dimensional flow, the third invariant $|e_{ik}|$ is identically zero. Hence η can be expressed as a scalar function of the second invariant e only in three-dimensional flow. Taking account of eq. (8), the second invariant e becomes

$$e = -\tfrac{1}{4}[\,2(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + e_{yz}^2 + e_{zx}^2 + e_{xy}^2\,] \qquad (9)$$

If we generalize Waele-Ostwald's formula by assuming that the stress components are, in general, expressible by the 1/n th power functions of the rate-of-strain components for non-Newtonian fluids, the relations between the stress components and the rate-of-strain components may be expressed as

$$p_{ii} = -p + 2\mu_{psu}^{1/n}\, e_{ii}\,\Theta, \quad (i = x, y, z)$$
$$p_{ik} = \mu_{psu}^{1/n}\, e_{ik}\,\Theta, \quad (i,k = x, y, z;\ i \neq k) \qquad (10)$$

$$\Theta = [\,2(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + e_{yz}^2 + e_{zx}^2 + e_{xy}^2\,]^{(1-n)/2n} \qquad (11)$$

If we include the kinetic reactions and the external forces, and consider the equilibrium of an infinitesimal rectangular parallelepiped with its centroid at a point P(x, y, z,) and its edges parallel to the coordinate axes, then by resolving the resultant forces along the axes, we obtain

$$\rho \frac{Du}{Dt} = \rho X + \frac{\Delta p_{xx}}{\Delta x} + \frac{\Delta p_{xy}}{\Delta y} + \frac{\Delta p_{xz}}{\Delta z}$$
$$\rho \frac{Dv}{Dt} = \rho Y + \frac{\Delta p_{yx}}{\Delta x} + \frac{\Delta p_{yy}}{\Delta y} + \frac{\Delta p_{yz}}{\Delta z} \qquad (12)$$
$$\rho \frac{Dw}{Dt} = \rho Z + \frac{\Delta p_{zx}}{\Delta x} + \frac{\Delta p_{zy}}{\Delta y} + \frac{\Delta p_{zz}}{\Delta z}$$

where X, Y, Z are the components of the extetnal force actin- on the fluid per unit mass, ρ is the density, and Du/Dt, Dv/Dt, and Dw/Dt are the components of the acceleration of a particle of the fluid. Substituting eq. (10) in eq. (12), then making use of eq (8) we have the fundamental equations of motion for non-Newtonian fluid in the form

$$\rho \frac{Du}{Dt} = \rho X - \frac{\Delta p}{\Delta x} + \mu_{psu}^{1/n} [\, \Theta \nabla^2 u + 2\, e_{xx} \frac{\Delta \Theta}{\Delta x} + e_{xy} \frac{\Delta \Theta}{\Delta y} + e_{xz} \frac{\Delta \Theta}{\Delta z}]$$
$$\rho \frac{Dv}{Dt} = \rho Y - \frac{\Delta p}{\Delta y} + \mu_{psu}^{1/n} [\, \Theta \nabla^2 v + e_{yx} \frac{\Delta \Theta}{\Delta x} + 2e_{yy} \frac{\Delta \Theta}{\Delta y} + e_{yz} \frac{\Delta \Theta}{\Delta z}] \qquad (13)$$
$$\rho \frac{Dw}{Dt} = \rho Z - \frac{\Delta p}{\Delta z} + \mu_{psu}^{1/n} [\, \Theta \nabla^2 w + e_{zx} \frac{\Delta \Theta}{\Delta x} + e_{zy} \frac{\Delta \Theta}{\Delta y} + 2e_{zz} \frac{\Delta \Theta}{\Delta z}]$$

where $\nabla^2 \equiv \frac{\Delta^2}{\Delta x^2} + \frac{\Delta^2}{\Delta y^2} + \frac{\Delta^2}{\Delta z^2}$

The equation of continuity is $e_{xx} + e_{yy} + e_{zz} = 0 \qquad (8')$

This is treating the fluid as being incompressible. Parameters should be theorized for variation in density for fluid under pressure and eqs. (12) and (13) should be solved with the appropriate terms inserted. This will be investigated in another paper.

If n = 1, eq. (13) becomes the Navier-Stokes equation for Newtonian fluid. The rheological constant n has the different values for varied fluids and n, in general, seldom has an integral value. The approximate solution by the energy method may be applied for non-Newtonian flow. The dissipation of energy in non-Newtonian flow may be calculated by

$$I = \iiint [e_{xx}p_{xx} + e_{yy}p_{yy} + e_{zz}p_{zz} + e_{yz}p_{yz} + e_{zx}p_{zx} + e_{xy}p_{xy}]\,[dxdydz]$$

Substituting eq. (10) into this equation and then using eq (8') we get

$$I = \mu_{psu}^{1/n} \iiint [2(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + e_{yz}^2 + e_{zx}^2 + e_{xy}^2]^{(n+1)/2n}\, dxdydz \qquad (14)$$

so that the dissipation function ϕ, which is the rate of dissipation of energy per unit time per unit volume is obtained as

$$\phi = \mu_{psu}^{1/n} [2(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + e_{yz}^2 + e_{zx}^2 + e_{xy}^2]^{(n+1)/2n} \qquad (15)$$

Now if the velocity distribution of flow gets the minimum dissipation of energy, that is $\delta I = \delta \int\int\int \phi \, dxdydz = 0$ (16)

It may be proved that the velocity distribution also satisfies the equation of motion in which inertia terms are neglected and external forces derived from a potential or zero, that is

$$\Theta \nabla^2 u + 2e_{xx} \frac{\Delta\Theta}{\Delta x} + e_{xy} \frac{\Delta\Theta}{\Delta y} + e_{xz} \frac{\Delta\Theta}{\Delta z} = \frac{\Delta H}{\Delta x}$$

$$\Theta \nabla^2 v + e_{yx} \frac{\Delta\Theta}{\Delta x} + 2e_{yy} \frac{\Delta\Theta}{\Delta y} + e_{yz} \frac{\Delta\Theta}{\Delta z} = \frac{\Delta H}{\Delta y} \qquad (17)$$

$$\Theta \nabla^2 w + e_{zx} \frac{\Delta\Theta}{\Delta x} + e_{zy} \frac{\Delta\Theta}{\Delta y} + 2e_{zz} \frac{\Delta\Theta}{\Delta z} = \frac{\Delta H}{\Delta z}$$

Where H is a function of x, y, z. From eq. (15) we have

$$\delta\phi = \frac{n+1}{n} \mu_{psu}^{1/n} \Theta[2\frac{\Delta u}{\Delta x}\frac{\Delta\delta u}{\Delta x} + 2\frac{\Delta v}{\Delta y}\frac{\Delta\delta v}{\Delta y} + 2\frac{\Delta w}{\Delta z}\frac{\Delta\delta w}{\Delta z} + (\frac{\Delta w}{\Delta y} + \frac{\Delta v}{\Delta z})(\frac{\Delta\delta w}{\Delta y} + \frac{\Delta\delta v}{\Delta z})$$
$$+ (\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})(\frac{\Delta\delta u}{\Delta z} + \frac{\Delta\delta w}{\Delta x}) + (\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y})(\frac{\Delta\delta v}{\Delta x} + \frac{\Delta\delta u}{\Delta y})]$$

Therefore the variation δI of I becomes

$$\delta I = \frac{n+1}{n} \mu_{psu}^{1/n} \int\int\int \Theta[2\frac{\Delta u}{\Delta x}\frac{\Delta\delta u}{\Delta x} + 2\frac{\Delta v}{\Delta y}\frac{\Delta\delta v}{\Delta y} + 2\frac{\Delta w}{\Delta z}\frac{\Delta\delta w}{\Delta z} + (\frac{\Delta w}{\Delta y} + \frac{\Delta v}{\Delta z})$$
$$(\frac{\Delta\delta w}{\Delta y} + \frac{\Delta\delta v}{\Delta z}) + (\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})(\frac{\Delta\delta u}{\Delta z} + \frac{\Delta\delta w}{\Delta x}) + (\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y})(\frac{\Delta\delta v}{\Delta x} + \frac{\Delta\delta u}{\Delta y})] \, dxdydz \qquad (18)$$

Considering that δu, δv, δw are zero at the body surface and infinity and then using the integration by parts, the differential terms of δu, δv, and δw with respect to x become

$$\int\int\int \Theta[2\frac{\Delta u}{\Delta x}\frac{\Delta\delta u}{\Delta x} + (\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y})\frac{\Delta\delta v}{\Delta x} + (\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})\frac{\Delta\delta w}{\Delta x}]dxdydz$$

$$= \int\int\Big\{|\Theta[2\frac{\Delta u}{\Delta x}\delta u + (\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y})\delta v + (\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})\delta w]| - \int \frac{\Delta\Theta}{\Delta x}[2\frac{\Delta u}{\Delta x}\delta u + (\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y})\delta v$$
$$+ (\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})\delta w]dx - \int \Theta\Big[2\frac{\Delta^2 u}{\Delta x^2}\delta u + \frac{\Delta}{\Delta x}(\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y})\delta v + (\frac{\Delta}{\Delta x})(\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})\delta w\Big]dx\Big\} dydz$$

$$= -\int\int\int \frac{\Delta\Theta}{\Delta x}[2\delta u\frac{\Delta u}{\Delta x} + \delta v(\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}) + \delta w(\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})] \, dxdydz$$
$$- \int\int\int \Theta \, [2\delta u \frac{\Delta^2 u}{\Delta x^2} + \delta v \frac{\Delta}{\Delta x}\{\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}) + \delta w \frac{\Delta}{\Delta x}(\frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x})] dxdydz$$

Since the differential terms of δu, δv, and δw wtih respect to y and z are transformed in the same way, eq. 18 may be written as

$$\delta I = -\frac{(n+1)}{n} \mu_{psu}^{1/n} \int\int\int [\delta u(\Theta\nabla^2 u + 2e_{xx}\frac{\Delta\Theta}{\Delta x} + e_{xy}\frac{\Delta\Theta}{\Delta y} + e_{xz}\frac{\Delta\Theta}{\Delta z}) + \delta v(\Theta\nabla^2 v$$
$$+ e_{xy}\frac{\Delta\Theta}{\Delta x} + 2e_{yy}\frac{\Delta\Theta}{\Delta y} + e_{yz}\frac{\Delta\Theta}{\Delta z}) + \delta w(\Theta\nabla^2 w + e_{zx}\frac{\Delta\Theta}{\Delta x} + e_{zy}\frac{\Delta\Theta}{\Delta y} + 2e_{zz}\frac{\Delta\Theta}{\Delta z})$$

$$+ \alpha\left(\delta u\frac{\Delta}{\Delta x} + \delta w \frac{\Delta}{\Delta z} + \delta v\frac{\Delta}{\Delta y}\right) \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} + \frac{\Delta w}{\Delta z}\right) \Big] dxdydz$$

Utilizing eq. (8') and eq. (16) the above equation becomes

$$\iiint \Big[\delta u\left(\Theta \nabla^2 u + 2e_{xx}\frac{\Delta\Theta}{\Delta x} + e_{xy}\frac{\Delta\Theta}{\Delta y} + e_{xz}\frac{\Delta\Theta}{\Delta z}\right) + \delta v\left(\Theta \nabla^2 v + e_{yx}\frac{\Delta\Theta}{\Delta x} + 2e_{yy}\frac{\Delta\Theta}{\Delta y} + e_{yz}\frac{\Delta\Theta}{\Delta z}\right) + \delta w\left(\Theta \nabla^2 w + e_{zx}\frac{\Delta\Theta}{\Delta x} + e_{zy}\frac{\Delta\Theta}{\Delta y} + 2e_{zz}\frac{\Delta\Theta}{\Delta z}\right)\Big] dxdydz = 0 \quad \underline{(19)}$$

Since the variations δu, δv, and δw must be satisfied by eq. (8) so that

$$\frac{\Delta\delta u}{\Delta x} + \frac{\Delta\delta v}{\Delta y} + \frac{\Delta\delta w}{\Delta z} = 0$$

Multiplying this equation by $\alpha(x, y, z)$ and then adding its results to eq. (19) produces the following equation: (20)

$$\iiint \Big[\delta u\left(\Theta \nabla^2 u + 2e_{xx}\frac{\Delta\Theta}{\Delta x} + e_{xy}\frac{\Delta\Theta}{\Delta y} + e_{xz}\frac{\Delta\Theta}{\Delta z}\right) + \delta v\left(\Theta \nabla^2 v + e_{yx}\frac{\Delta\Theta}{\Delta x} + 2e_{yy}\frac{\Delta\Theta}{\Delta y} + e_{yz}\frac{\Delta\Theta}{\Delta z}\right) + \delta w\left(\Theta \nabla^2 w + e_{zx}\frac{\Delta\Theta}{\Delta x} + e_{zy}\frac{\Delta\Theta}{\Delta y} + 2e_{zz}\frac{\Delta\Theta}{\Delta z}\right) + \alpha\left(\frac{\Delta\delta u}{\Delta x} + \frac{\Delta\delta v}{\Delta y} + \frac{\Delta\delta w}{\Delta z}\right)\Big] dxdydz$$

equals zero. Since δu, δv, and δw are zero at the body surface and infinity, using integration by parts gives

$$\iiint \alpha \frac{\Delta}{\Delta x}\delta u\, dxdydz = \iint\Big[\,|\alpha\delta u| - \int \frac{\Delta\alpha}{\Delta x}\delta u\, dx\Big]\, dydz = -\iiint \frac{\Delta\alpha}{\Delta x}\delta u\, dxdydz$$

Therefore eq. (20) becomes

$$\iiint \Big[\left(\Theta \nabla^2 u + 2e_{xx}\frac{\Delta\Theta}{\Delta x} + e_{xy}\frac{\Delta\Theta}{\Delta y} + e_{xz}\frac{\Delta\Theta}{\Delta z} - \frac{\Delta\alpha}{\Delta x}\right)\delta u + \left(\Theta \nabla^2 v + e_{yx}\frac{\Delta\Theta}{\Delta x} + 2e_{yy}\frac{\Delta\Theta}{\Delta y} + e_{yz}\frac{\Delta\Theta}{\Delta z} - \frac{\Delta\alpha}{\Delta y}\right)\delta v + \left(\Theta \nabla^2 w + e_{zx}\frac{\Delta\Theta}{\Delta x} + e_{zy}\frac{\Delta\Theta}{\Delta y} + 2e_{zz}\frac{\Delta\Theta}{\Delta z} - \frac{\Delta\alpha}{\Delta z}\right)\delta w\Big] dxdydz = 0$$

As δu, δv, and δw may take arbitrary values in the domain of flow, the coefficients of δu, δv, and δw must be zero in order to always satisfy the above equation. Then, the same equation as eq. 17 is obtained as follows:

$$\Theta \nabla^2 u + 2e_{xx}\frac{\Delta\Theta}{\Delta x} + e_{xy}\frac{\Delta\Theta}{\Delta y} + e_{xz}\frac{\Delta\Theta}{\Delta z} = \frac{\Delta\alpha}{\Delta x}$$

$$\Theta \nabla^2 v + e_{yx}\frac{\Delta\Theta}{\Delta x} + 2e_{yy}\frac{\Delta\Theta}{\Delta y} + e_{yz}\frac{\Delta\Theta}{\Delta z} = \frac{\Delta\alpha}{\Delta y}$$

$$\Theta \nabla^2 w + e_{zx}\frac{\Delta\Theta}{\Delta x} + e_{zy}\frac{\Delta\Theta}{\Delta y} + 2e_{zz}\frac{\Delta\Theta}{\Delta z} = \frac{\Delta\alpha}{\Delta z}$$

Thus, the velocity distribution that satisfies eq. (16) is found to be the solution of eq. (17), if the inertia terms are neglected and external forces derived from a potential or zero. Therefore, if the flow satisfies the conditions mentioned above, the approximate solution of flow may be obtained by using this minimum dissipation of energy method: the velocity distribution containing some undetermined coefficients and satisfying the boundary conditions is appropriately assumed, and then the undetermined coefficients are determined so that the dissipation energy becomes minimum.

VISCO-ELASTICITY

Some of the characteristics of visco-elastic substances are desirable in the operation of hydraulic systems of aircraft, weapon systems and transportation equipment. In effect, there is a need to absorb energy without a shock result. In visco-elastic substances, the ratio of lateral to longitudinal strain in uni-axial stressing is not a constant but a variable parameter depending on time and the conditions of the test. A theoretical analysis of this parameter under various assumptions concerning the visco-elastic response of the material and the testing conditions as developed by Freudenthal and others is given. The Maxwell body more closely approximates the desired reaction. Promise of further approximation is shown in the characteristics of some long chain polymeric fluids which show a high bulk modulus under high pressures. The final solution to the problem will be accomplished by development of the desirable characteristics from a molecular structure approach.

The stress and strain components σ_{ij} and ε_{ij} may be written as S_{ij}, e_{ij} and σ_{kk}, ε_{kk}, respectively, so that

$$\sigma_{ij} = S_{ij} + 1/3\,\delta_{ij}\sigma_{kk} \quad \text{and} \quad \varepsilon_{ij} = e_{ij} + 1/3\,\delta_{ij}\,\varepsilon_{kk} \tag{1.1}$$

where δ_{ij} is the same Kronecker delta. The general linear visco-elastic medium is defined by the linear operator equations

$$\mathbb{P}\,S_{ij} = 2G\mathbb{Q}\,e_{ij} \quad \text{and} \quad \mathbb{P}'\sigma_{kk} = 3K\mathbb{Q}'(\varepsilon_{kk} - 3\alpha T) \tag{1.2}$$

where T denotes the temperature difference with respect to a constant reference temperature and α the coefficient of thermal expansion. The constants of the operators

$$\begin{aligned} \mathbb{P} &= a_0 + a_1\frac{\Delta}{\Delta t} + \cdots\cdots a_m\frac{\Delta^m}{\Delta t^m} \\ \mathbb{Q} &= b_0 + b_1\frac{\Delta}{\Delta t} + \cdots\cdots b_n\frac{\Delta^n}{\Delta t^n} \\ \mathbb{P}' &= a'_0 + a'_1\frac{\Delta}{\Delta t}\cdots\cdots + a'_r\frac{\Delta^r}{\Delta t^r} \\ \mathbb{Q}' &= b'_0 + b'_1\frac{\Delta}{\Delta t} + \cdots\cdots b'_s\frac{\Delta^s}{\Delta t^s} \end{aligned} \tag{1.3}$$

are combinations of relaxation times, retardation times and shear moduli.

Combining eqs. (1.2) under the assumption of isothermal conditions, the visco-elastic stress-strain relations may be obtained in the alternative forms:

$$\sigma_{ij} = \left(3K\frac{\mathbb{Q}'}{\mathbb{P}'} - 2G\frac{\mathbb{Q}}{\mathbb{P}}\right)(1/3)\varepsilon_{kk}\,\delta_{ij} + 2G\frac{\mathbb{Q}}{\mathbb{P}}\,\varepsilon_{ij}$$

and

$$\varepsilon_{ij} = \left(\frac{1}{9K}\frac{\mathbb{P}'}{\mathbb{Q}'} - \frac{1}{6G}\frac{\mathbb{P}}{\mathbb{Q}}\right)\sigma_{kk}\,\delta_{ij} + \frac{1}{2G}\frac{\mathbb{P}}{\mathbb{Q}}\,\sigma_{ij} \tag{1.4}$$

Poisson's ratio ν for the elastic medium related to K and G by the relation

$$\nu = (3K - 2G)/2(3K + G) \tag{1.5}$$

can be transformed and becomes, in linear visco-elastic theory, the

OPERATOR

$$\nu = \frac{3KPQ' - 2GQP'}{2(3KPQ' + GQP')} \qquad (1.6)$$

Dividing the numerator and denominator by 18 KG, eq. 1.6 takes the form

$$\nu = \frac{\frac{1}{9K}\frac{P'}{Q'} - \frac{1}{6G}\frac{P}{Q}}{\frac{1}{9K}\frac{P'}{Q'} + \frac{1}{3G}\frac{P}{Q}} \qquad (1.6a)$$

which can be obtained by evaluating the second eq. (1.4) for the condition. Thus

$$\varepsilon_{11} = \left(\frac{1}{9K}\frac{P'}{Q'} + \frac{1}{3G}\frac{P}{Q}\right)\sigma_{11}$$

and

$$\varepsilon_{22} = \left(\frac{1}{9K}\frac{P'}{Q'} - \frac{1}{6G}\frac{P}{Q}\right)\sigma_{11} = \varepsilon_{33} \qquad (1.7)$$

the time dependent ratio $\nu(t)$ is obtained as the ratio $\nu(t) = -\varepsilon_{22}/\varepsilon_{11} = \nu(\sigma_{11})$

Laplace transforms are used to evaluate this ratio by use of the stress-strain relaions (1.2) and (1.4) and the operator equations (1.3)

$$\bar{P}(p) = \sum_{k=1}^{k=m} a_k p^k \qquad \bar{Q}(p) = \sum_{k=1}^{k=n} b_k p^k$$

and

$$\bar{P}'(p) = \sum_{k=1}^{k=r} a'_k p^k \qquad \bar{Q}'(p) = \sum_{k=1}^{k=s} b'_k p^k \qquad (1.8)$$

The ratios

$$\frac{1}{\bar{E}}(p) = \frac{\bar{P}(p)}{\bar{Q}(p)} \quad \text{and} \quad \frac{1}{\bar{E}'}(p) = \frac{\bar{P}'(p)}{\bar{Q}'(p)} \qquad (1.9)$$

are quotients of two polynomials, the inverse transforms of which can usually be expr ssed in the form of series of negative exponentials.

The transformed eqs. (1.7) can be directly written in the form

$$\bar{\varepsilon}_{11} = \left(\frac{1}{9K\,\bar{E}'(p)} + \frac{1}{3G\,\bar{E}(p)}\right)\bar{\sigma}_{11}$$

and

$$\bar{\varepsilon}_{22} = \left(\frac{1}{9K\,\bar{E}'(p)} - \frac{1}{6G\,\bar{E}(p)}\right)\bar{\sigma}_{11} \qquad (1.10)$$

Poisson's ratio $\nu(t)$ is obtained as the inverse transform

$$\nu(t) = -\frac{L^{-1}\left(\left[\frac{1}{9K\,\bar{E}'(p)} - \frac{1}{6G\,\bar{E}(p)}\right]\bar{\sigma}_{11}\right)}{L^{-1}\left(\left[\frac{1}{9K\,\bar{E}'(p)} + \frac{1}{3G\,\bar{E}(p)}\right]\bar{\sigma}_{11}\right)} \tag{1.11}$$

When $\varepsilon_{11} = \varepsilon_{11}(t)$ and $\bar{\varepsilon}_{11}(p)$ is given, with $\sigma_{22} = \sigma_{33} = 0$, $\bar{\sigma}_{11}$ is evaluated with the aid of the first eq. (1.10) and introduced into the second. Hence the ratio

$$\nu(t) = L^{-1}\left\{\left[\frac{\frac{1}{9K\,\bar{E}'(p)} - \frac{1}{6G\,\bar{E}(p)}}{\frac{1}{9K\,\bar{E}'(p)} + \frac{1}{3G\,\bar{E}(p)}}\right]\bar{\varepsilon}_{11}\right\} \Big/ \varepsilon_{11}(t) \tag{1.12}$$

Eq. (1.11) and (1.12) will now be evaluated for the Maxwell type of idealized linear visco-elastic behavior in volume-constant distortion. If we assume elastic volumetric deformation $\bar{E}'$ reduces to unity. The testing conditions investigated are : constant stress σ_{11} = const., constant stress-rate $\sigma_{11} = ct$ where c is an arbitrary constant, constant strain ε_{11} = const. and constant strain-rate $\varepsilon_{11} = ct$. For the actual evaluation the arbitrary relation $3K = 8G$ has been introduced, for which the frequently used elastic value $\nu = 1/3$ is obtained.

Evaluation of "Poisson's Ratio for Simple Elastically Compressible Visco-Elastic Media--Maxwell Body

The mechanical behavior of a Maxwell body is determined by

$$\bar{P} = \frac{1}{\tau} + p \qquad \bar{Q} = p$$
$$\bar{P}' = 1 \qquad \bar{Q}' = 1$$

where $\tau = \eta / G$

Constant stress σ_{11} is specifically

$$\sigma_{11} = c\,H(t), \qquad \bar{\sigma}_{11} = c\,\frac{1}{p}$$

From eq. (1.11)

$$\nu(t) = \frac{L^{-1}\left[\left(\frac{1+\tau p}{6G\,\tau p} - \frac{1}{9K}\right)\frac{c}{p}\right]}{L^{-1}\left[\left(\frac{1+\tau p}{3G\,\tau p} + \frac{1}{9K}\right)\frac{c}{p}\right]}.$$

$$= \frac{1}{2} \frac{\frac{t}{\tau} + 1 - \frac{2G}{3K}}{\frac{t}{\tau} + 1 + \frac{G}{3K}}$$

The diagram of $\nu(t)$ for $G/K = 3/8$ is shown in figure 1, curve $\sigma_{11} = c$. At $t = 0$, the value of $\nu = 1/3$ and as $t \to \infty$, $\nu = 1/2$. These limits may be explained by the fact that the instantaneous response of a Maxwell body is purely elastic and consequently ν is controlled by the ratio G/K only. Under long time loading a Maxwell body behaves as a viscous fluid which results in $\nu \to 1/2$, with the elastic compressibility becoming unnoticeable under the uniaxial testing conditions.

The constant stress rate is specified by

$$\sigma_{11} = ct\ H(t) \qquad \text{and} \qquad \bar{\sigma}_{11} = c\,\frac{1}{p^2}$$

From eq. (1.11)

$$\nu(t) = \frac{L^{-1}\left[\left(\frac{1+\tau p}{6G\tau p} - \frac{1}{9K}\right)\frac{c}{p^2}\right]}{L^{-1}\left[\left(\frac{1+\tau p}{3G\tau p} + \frac{1}{9K}\right)\frac{c}{p^2}\right]} = \frac{\frac{t}{2\tau} + 1 - \frac{2G}{3K}}{2\left(\frac{t}{2\tau} + 1 + \frac{G}{3K}\right)}$$

The corresponding diagram is shown in figure 1, curve $\sigma_{11} = ct$

Constant strain is specified by

$$\varepsilon_{11} = c\ H(t) \qquad \text{and} \qquad \bar{\varepsilon}_{11} = c\,\frac{1}{p}$$

From eq. (1.12)

$$\nu(t) = L^{-1}\left[\frac{3K\left(\frac{1}{\tau} + p\right) - 2Gp}{2\left[3K\left(\frac{1}{\tau} + p\right) + Gp\right]p}\right]$$

$$= \frac{1}{2} + \left(\frac{3K - 2G}{3K + G} - 1\right)\exp\left(-\frac{3K}{3K + G}\,\frac{t}{\tau}\right) \qquad (1.13)$$

This is shown by curve $\varepsilon_{11} = c$ in figure 1

Constant strain rate is specified by

$$\varepsilon_{11} = c\ t\ H(t) \qquad \text{and} \qquad \bar{\varepsilon}_{11} = c\,\frac{1}{p^2}$$

From eq. (1.12)

$$\nu(t) = L^{-1}\left\{\frac{3K\left(\frac{1}{\tau} + p\right) - 2Gp}{2\left[3K\left(\frac{1}{\tau} + p\right) + Gp\right]p^2}\right\} \Big/ t \qquad (1.13)'$$

$$= \frac{1}{2} - \frac{G\,\tau}{2K\,t}\left[\,1 - \exp\left(-\frac{3\,K}{3\,K + G}\;\frac{t}{\tau}\right)\right] \qquad (1.14)$$

This is shown by curve $\varepsilon_{11} = c\ t$ in figure 1

But $\tau^{n} = \frac{\mu du}{dy}$

Then eq. (1.13) becomes

$$\nu\,(t) = \frac{1}{2} + \left(\frac{3\,K - 2\,G}{3\,K + G} - 1\right) \exp\left(-\frac{3\,K}{3\,K + G}\;\frac{t}{\left(\frac{\mu du}{dy}\right)^{1/n}}\right)$$

and eq. (1.14) becomes

$$\nu\,(t) = \frac{1}{2} - \frac{G\left(\frac{\mu du}{dy}\right)^{1/n}}{2\,K\,t}\left[\,1 - \exp\left(-\frac{3\,K}{3\,K + G}\;\frac{t}{\left(\frac{\mu du}{dy}\right)^{1/n}}\right)\right]$$

which gives the ratio of lateral to longitudinal strain in uni-axial stressing in terms of viscosity and fluid velocity.

The paper has proposed a generalization of the Navier-Stokes equation for non-Newtonian fluids in which the stress is proportional to a power n of the rate of strain. The ratio of the transverse to the longitudinal strain is not a constant but a function of time or of strain and depends strongly upon the visco-elastic response of the medium as well as on the loading conditions.

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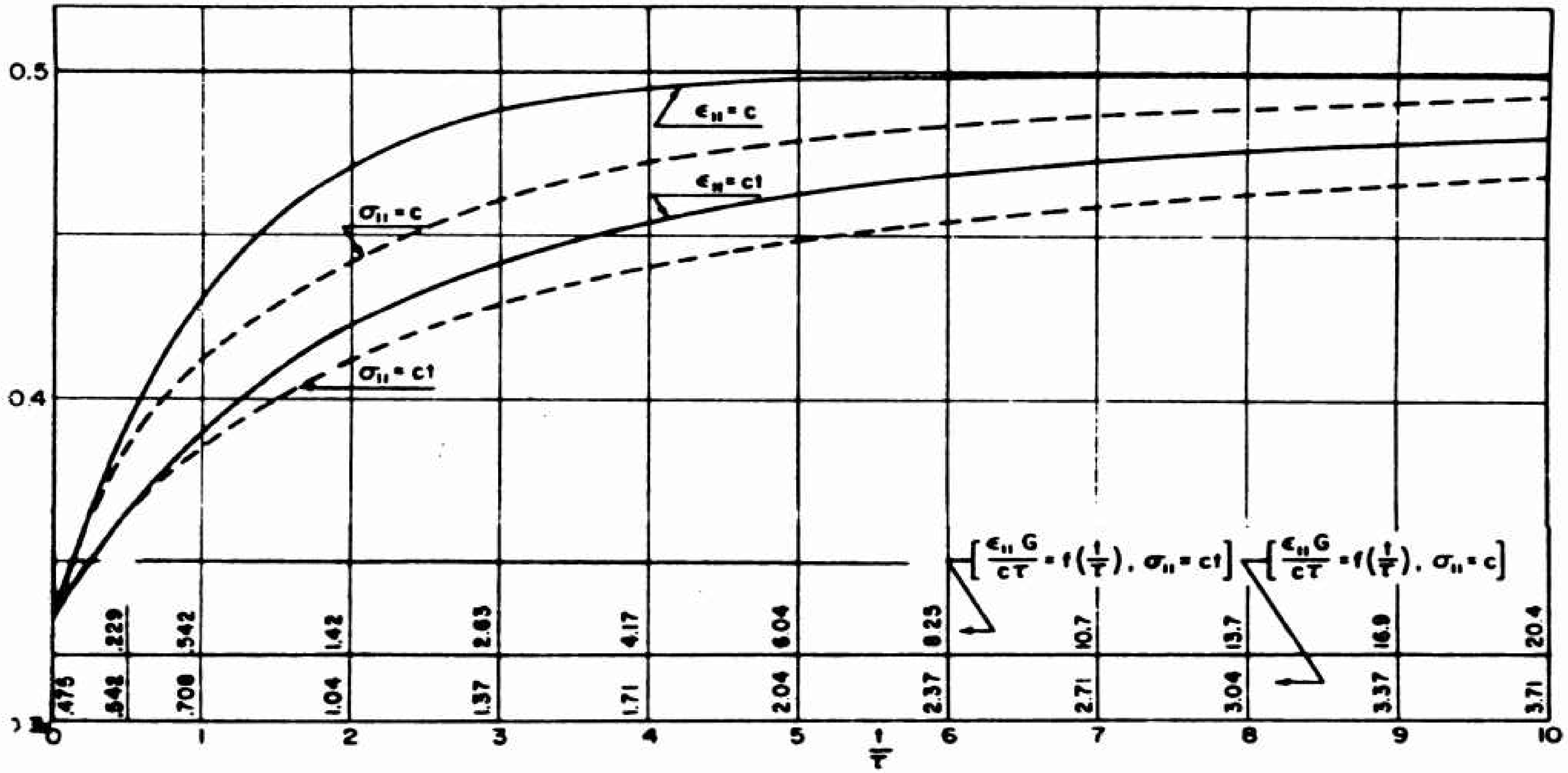

Fig. 1. Poisson's Ratio ν as a Function of Time $\nu(t/\tau)$ or of strain for Elastically Compressible Maxwell Body.